Naval Research Laboratory

Washington, DC 20375-5320

NRL/FR/7218--04--10,101

Rayleigh-Sommerfeld Diffraction vs Fresnel-Kirchhoff, Fourier Propagation, and Poisson's Spot

Robert L. Lucke

Radio/IR/Optical Sensors Branch
Remote Sensing Division

December 30, 2004

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

1. REPORT DATE *(DD-MM-YYYY)*
30-12-2004

2. REPORT TYPE
NRL Formal Report

3. DATES COVERED *(From - To)*
June 2003 to June 2004

4. TITLE AND SUBTITLE
Rayleigh-Sommerfeld Diffraction vs Fresnel-Kirchhoff, Fourier Propagation, and Poisson's Spot

5a. CONTRACT NUMBER
1251876

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER

5d. PROJECT NUMBER

5e. TASK NUMBER
JPL

5f. WORK UNIT NUMBER
8393

6. AUTHOR(S)
Robert L. Lucke

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Naval Research Laboratory
Washington, DC 20375-5320

8. PERFORMING ORGANIZATION REPORT NUMBER
NRL/FR/7218--04-10,101

9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES)
National Aeronautics and Space Administration
300 E St. S.W.
Washington, DC 20024-3210

Jet Propulsion Laboratory
4800 Oak Grove Dr.
Pasadena, CA 91109

10. SPONSOR / MONITOR'S ACRONYM(S)

11. SPONSOR / MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION / AVAILABILITY STATEMENT
Approved for public release; distribution is unlimited.

13. SUPPLEMENTARY NOTES

14. ABSTRACT
When the Fresnel-Kirchhoff (FK) diffraction integral is evaluated exactly (instead of using the Fresnel approximation), the well-known mathematical inconsistency in the FK boundary conditions leads to unacceptable results for the intensity of Poisson's spot. The Rayleigh-Sommerfeld (RS) integral doesn't have this problem. Fourier propagation is equivalent to RS diffraction and provides another means of deriving a diffraction integral.

15. SUBJECT TERMS
Diffraction
Fourier propagation
Poisson's spot

16. SECURITY CLASSIFICATION OF:			**17. LIMITATION OF ABSTRACT**	**18. NUMBER OF PAGES**	**19a. NAME OF RESPONSIBLE PERSON**
a. REPORT	**b. ABSTRACT**	**c. THIS PAGE**	UL	17	Dr. Robert L. Lucke
Unclassified	Unclassified	Unclassified			**19b. TELEPHONE NUMBER** *(include area code)* (202) 767-2749

Standard Form 298 (Rev. 8-98)
Prescribed by ANSI Std. Z39.18

CONTENTS

RAYLEIGH-SOMMERFELD DIFFRACTION VS FRESNEL-KIRCHHOFF, FOURIER PROPAGATION, AND POISSON'S SPOT

1. INTRODUCTION

The boundary conditions imposed on the diffraction problem in order to obtain the Fresnel-Kirchhoff (FK) solution are well-known to be mathematically inconsistent and to be violated by the solution when the observation point is close to the diffracting screen [1-3]. These problems are absent in the Rayleigh-Sommerfeld (RS) solution. The difference between RS and FK is in the inclination factor and is usually immaterial because the inclination factor is approximated by unity. But when this approximation is not valid, FK can lead to unacceptable answers. Calculating the on-axis intensity of Poisson's spot provides a critical test, a test passed by RS and failed by FK. FK fails because (a) convergence of the integral depends on how it is evaluated and (b) when the convergence problem is fixed, the predicted amplitude at points near the obscuring disk is not consistent with the assumed boundary conditions.

Poisson's spot, also known as the spot of Arago, is the name given to the bright on-axis spot behind a circular obscuration illuminated by a plane wave: on the axis of the disk, all light diffracted at the rim of the disk arrives in phase and interferes constructively. Consequently, even for angles approaching 90°, i.e., for observation points close to the disk, diffraction can result in a significant intensity. (In the overwhelming majority of optics problems, diffraction at angles of more than a few degrees leads to vanishingly small intensities and can be safely ignored.) Treatments of Poisson's spot that make the Fresnel approximation [4,5] do not apply to the region close to the disk. Treatments that do apply close to the disk use the RS integral [6] or its predecessor, the Rayleigh integral [7]. The RS integral is derived from the Rayleigh integral with the additional assumption that the wavelength of the light is small compared to the geometric dimensions of the problem, an assumption that is made throughout this report. I refer to a calculation as exact to indicate that, while the short-wavelength approximation has been made, the Fresnel approximation has not. The other fundamental approximation made here is the scalar wave approximation, which means that the results apply better to acoustics than to optics, a point that will be reconsidered in the concluding remarks of Section 5.

Fourier propagation provides an alternate means of handling diffraction problems. It can be used in computer software to calculate beam propagation problems, but in this report, Fourier propagation theory is used to derive solutions in integral form for the 2-D (long slits and strips) and 3-D (arbitrarily shaped apertures) diffraction problems. Standard Fourier propagation reproduces the RS diffraction integral.

2. THE BASIC DIFFRACTION PROBLEM

Solving the basic diffraction problem requires finding a solution to the Helmholtz equation for a propagating wave encountering a partially obscuring planar screen. The Helmholtz equation is

$$\left(\nabla^2 + k^2\right) U(x,y,z) = 0\,, \tag{1}$$

Manuscript approved September 13, 2004

where $k = 2\pi/\lambda$ and U describes the amplitude and phase of the wave. U is a scalar, so only scalar diffraction theory is addressed. The boundary condition imposed on the solution to this differential equation is the effect of a diffracting screen in the $z = 0$ plane. Denoting by T the parts of the screen that are transmissive and by B the parts that block the beam, the boundary conditions used for the RS and FK solutions are

$$\begin{array}{lll} \text{RS and FK:} & U(x,y,0) = U_0(x,y,0) & \text{for } (x,y) \in T, \\ & U(x,y,0) = 0 & \text{for } (x,y) \in B, \\ \text{FK only:} & \left.\dfrac{\partial U(x,y,z)}{\partial z}\right|_{z=0} = \left.\dfrac{\partial U_0(x,y,z)}{\partial z}\right|_{z=0} & \text{for } (x,y) \in T, \\ & \left.\dfrac{\partial U(x,y,z)}{\partial z}\right|_{z=0} = 0 & \text{for } (x,y) \in B, \end{array} \tag{2}$$

where U_0 describes the incident wave and $\partial U/\partial z$ is the derivative of U normal to the diffracting plane.

With reference to Fig. 1, the RS diffraction integral [1,2] for U at distance z behind an aperture in a planar mask is

$$U_{RS}(x', y', z) = -\frac{i}{\lambda} \int_{Area} U_0(x,y,0) \frac{\exp(ikr)}{r} \cos \chi dxdy, \tag{3}$$

where $r = [(x' - x)^2 + (y' - y)^2 + z^2]^{1/2}$ is the distance between $(x, y, 0)$ and (x', y', z), χ is the diffraction angle at point $(x, y, 0)$, i.e., the angle the diffracted ray makes with the normal to the plane (not with the direction of the incoming wave) and the integral is over the area of the aperture. The FK integral [1,3] is

$$U_{FK}(x', y', z) = -\frac{i}{\lambda} \int_{Area} U_0(x,y,0) \frac{\exp(ikr)}{r} \frac{1}{2} (\cos \zeta + \cos \chi) dxdy, \tag{4}$$

where ζ is the incidence angle at point $(x, y, 0)$. The $\exp(ikr)/r$ factors in Eqs. (3) and (4) express Huygens' principle: each point in the aperture acts as a source of spherical waves that combine to give the diffraction pattern. The cosine factors are called the inclination factors and constitute the only difference between RS and FK. For most diffraction problems, the inclination factor is approximated by unity, causing the difference between RS and FK to disappear.

Goodman [1] gives a succinct derivation of both RS and FK and discusses the difference between them. The derivations use different Green functions, which require different boundary conditions to reduce the Green's theorem integral to the familiar diffraction integrals given in Eqs. (3) and (4). The basic problem with FK is that the Green theorem integral can't be evaluated unless the values of both U and $\partial U/\partial z$ are assumed

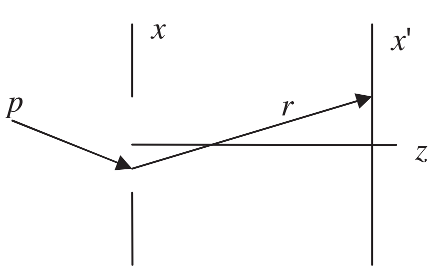

Fig. 1 — The basic diffraction problem: a point source p illuminates an aperture and produces a diffraction pattern on an observation screen. The distance r is the distance from a point in the aperture to a point on the screen. The angles ζ and χ (not labeled in the figure) are the angles the incident and the diffracting rays make with the z axis.

to be zero on the obscuring part of the diffracting screen, but we know from analytic function theory that if both U and $\partial U/\partial z$ are zero over any region, then $U \equiv 0$ everywhere. Thus, the FK solution cannot be fully mathematically consistent and must therefore be suspected of not always (at least) giving the right answer. In Section 3, I use Poisson's spot to show explicitly that the FK solution can violate the boundary conditions that were assumed for its derivation.

3. CALCULATING THE INTENSITY OF POISSON'S SPOT

For the simple case of a plane wave impinging normally on a circular disk of radius a (Fig. 2), $\cos\zeta = 1$ and $\cos\chi = z/r$. I calculate the on-axis amplitude and intensity behind the disk by doing so for an annular aperture and letting the outer radius of the annulus approach infinity:

$$U(0,0,z) = -\frac{i}{\lambda} U_0 \int_{\rho=a}^{\rho\to\infty} \frac{\exp(ikr)}{r} \frac{1}{2}\left(c_1 + c_2 \frac{z}{r}\right) 2\pi\rho d\rho, \tag{5}$$

where $c_1 = 0$, $c_2 = 2$ for RS; $c_1 = c_2 = 1$ for FK; and ρ is the radial coordinate in the (x, y) plane. In order to evaluate Eq. (5) without using the Fresnel approximation, I follow Sommerfeld [2] and Harvey et al. [6] in changing the variable of integration from ρ to r. On the z axis, $r = (z^2 + \rho^2)^{1/2}$, so $a \le \rho \le \infty \Rightarrow r_0 \le r \le \infty$, where $r_0 = (z^2 + a^2)^{1/2}$. Now $r^2 = z^2 + \rho^2$, so $rdr = \rho d\rho$, and the integral in Eq. (5) can be put in the right form to be evaluated via the Sommerfeld lemma given in Appendix A:

$$\begin{aligned}
U(0,0,z) &= -ikU_0 \int_{r_0}^{r\to\infty} \exp(ikr) \frac{1}{2}\left(c_1 + c_2 \frac{z}{r}\right) dr \\
&= -ikU_0 \left[\frac{1}{ik} \exp(ikr) \frac{1}{2}\left(c_1 + c_2 \frac{z}{r}\right) \Bigg|_{r=r_0}^{r\to\infty} + O\left(\frac{1}{k^2}\right) \right] \\
&\approx U_0 \frac{1}{2}\left(c_1 + c_2 \frac{z}{r_0}\right) \exp(ikr_0) - U_0 \frac{c_1}{2} \exp[ik(r\to\infty)].
\end{aligned} \tag{6}$$

The RS version of Eq. (6) is

$$U_{RS}(0,0,z) = U_0 \frac{z}{r_0} \exp(ikr_0), \tag{7}$$

while the FK version is

$$U_{FK}(0,0,z) = U_0 \frac{1}{2}\left(1 + \frac{z}{r_0}\right) \exp(ikr_0) - U_0 \frac{1}{2} \exp[ik(r\to\infty)], \tag{8}$$

which shows that the FK integral fails to converge in this case. The reader's attention is called to the fact that if a is set to zero (i.e., $r_0 = z$), there is no obscuration and the right side of Eq. (8) should be just $U_0\exp(ikz)$ – which, because of the second term, it isn't! The FK integral, when evaluated in this straightforward way, doesn't give an acceptable answer.

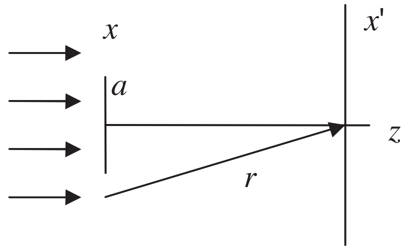

Fig. 2 — The diffraction problem for Poisson's spot; a plane wave falls on a circular disk. The distance $r = (x^2 + y^2 + z^2)^{1/2}$ is the distance from a point in the plane of diffraction to a point on the axis of the disk.

Eliminating the second term in Eq. (8) can be done in various ways. An artificial way would be to make the outer edge of the annular aperture an ellipse instead of a circle. Then the rays diffracted from this edge do not arrive on the axis in phase and do not interfere constructively. But the most sensible way is to impose Babinet's principle as a separate requirement (separate, because, as we have just seen, FK doesn't satisfy it unless the integral is done the right way). Babinet's principle requires that the sum of the obscuration diffraction pattern (Fig. 2) and the aperture diffraction pattern (Fig. 2 with the transmitting and blocking regions reversed) be the uninterrupted plane wave: $U_{ob} + U_{ap} = U_0\exp(ikz)$. Thus $U_{ob} = U_0\exp(ikz) - U_{ap}$, which can be seen from Eqs. (5) and (6) to be Eq. (8) without the second term:

$$\begin{aligned} U_{FK}(0,0,z) &= U_0 \exp(ikz) + \frac{i}{\lambda} U_0 \int_0^a \frac{\exp(ikr)}{r} \frac{1}{2}\left(1+\frac{z}{r}\right) 2\pi\rho \, d\rho \\ &= U_0 \frac{1}{2}\left(1+\frac{z}{r_0}\right) \exp(ikr_0). \end{aligned} \tag{9}$$

The remaining, and more serious, problem with Eq. (9) is that it doesn't satisfy the boundary condition under which it was derived. As $z \to 0$, we should find $U(0, 0, z) \to 0$ in Eqs. (7) and (9), since, from Eq. (2), that is the boundary condition originally assumed. Eq. (7) satisfies this condition, while Eq. (9) doesn't. Also, omitting the $\exp(ika)$ phase factor, $\partial U/\partial z = U_0/a$ at $z = 0$ for RS and half this for FK. The non-zero value of $\partial U/\partial z$ is consistent with RS, for which only U need be zero at the disk, but not for FK, which began with the additional boundary condition $\partial U/\partial z = 0$.

Eqs. (7) and (9) are squared to obtain intensity and are plotted in Fig. 3. The predicted intensities begin to differ appreciably at $z/a \approx 4$, where the diffraction angle is $\chi \approx 15°$. Note that the exactly on-axis intensity of Poisson's spot doesn't depend on wavelength because there is no wavelength dependence in the inclination factor. Wavelength dependence enters in the radial intensity distribution, which, for the central region defined by $r_\perp/a << 1$, where $r_\perp$ is the 2-D radius measured from the z axis, can be shown to be [4]

$$U(r_\perp, z) = U(0,0,z) \exp\left(\frac{i\pi r_\perp^2}{\lambda z}\right) J_0\left(\frac{2\pi a r_\perp}{\lambda z}\right). \tag{10}$$

The first zero of J_0 is at $2\pi a r_\perp/\lambda z = 2.4$, which shows that the diameter of Poisson's spot is proportional to λ, so the spot's area and the power contained in it are proportional to λ^2.

4. DOING DIFFRACTION WITH FOURIER PROPAGATION

In Section 4.1, I invoke the basic principles of Fourier propagation, and then use these principles in Section 4.2 to derive the RS diffraction integral. For ease of presentation, I address the 2-D diffraction problem: long slits or strips illuminated by plane or cylindrical waves that can be described by $U(x, z)$. In Section 4.1 the generalization to 3-D is obvious; for Section 4.2, it is done in Appendix B.

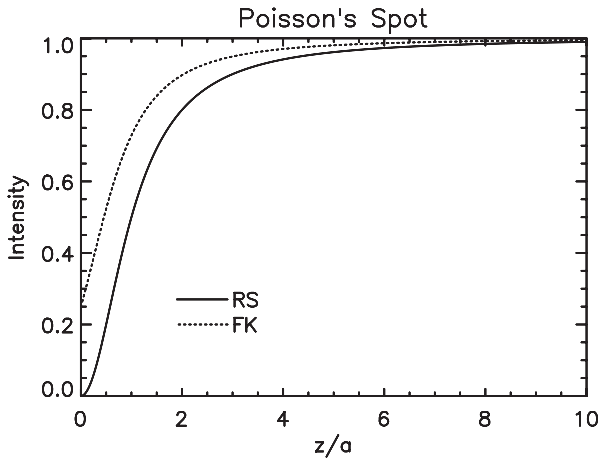

Fig. 3 — Intensity of Poisson's spot at on-axis distance z behind a disk of radius a, according to the Rayleigh-Sommerfeld and Fresnel-Kirchhoff diffraction theories.

4.1 Fourier Propagation

The Fourier transform over x of $U(x, z)$ is called its angular spectrum, defined by

$$A(\alpha, z) \equiv \int_{-\infty}^{\infty} U(x, z) \exp(-ik\alpha x) dx. \tag{11}$$

Fourier propagation rests on the premise that, as shown by Goodman [1] for example,

$$A(\alpha, z) = A(\alpha, 0) \exp\left(ik\sqrt{1-\alpha^2}\, z\right), \tag{12}$$

where α is not restricted to $-1 \le \alpha \le 1$. The Fourier transform variable in Eq. (11) is α/λ, so the inverse transform of $A(\alpha, z)$ is

$$\begin{aligned} U(x', z) &= \int_{-\infty}^{\infty} A(\alpha, z) \exp(ik\alpha x') d\left(\frac{\alpha}{\lambda}\right) \\ &= \int_{-\infty}^{\infty} A(\alpha, 0) \exp\left(ik\sqrt{1-\alpha^2}\, z + ik\alpha x'\right) d\left(\frac{\alpha}{\lambda}\right) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(x, 0) \exp(-ik\alpha x) dx \exp\left(ik\sqrt{1-\alpha^2}\, z + ik\alpha x'\right) d\left(\frac{\alpha}{\lambda}\right) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} U(x, 0) \exp\left[ik\sqrt{1-\alpha^2}\, z + ik\alpha(x' - x)\right] dx d\left(\frac{\alpha}{\lambda}\right). \end{aligned} \tag{13}$$

The second equality shows that $U(x', z)$ is the sum of plane waves of amplitude $A(\alpha, 0)$, propagating at angle $\theta = \cos^{-1}\alpha$ with respect to the x axis. When $|\alpha| > 1$, the plane waves are evanescent, with exponentially decaying z-dependence given by $\exp[-(\alpha^2 - 1)^{1/2} z]$. These waves do not propagate a significant distance from the aperture but are needed to give a complete Fourier decomposition of $U(x', z)$ at, or near, $z = 0$.

Since the dependence on x' and z is all in the exponent on the right side of Eq. (13), it is easy to see that

$$\left(\frac{\partial^2}{\partial x'^2}+\frac{\partial^2}{\partial z^2}\right)U(x',z)=\left[(ik\alpha)^2+\left(ik\sqrt{1-\alpha^2}\right)^2\right]U(x',z)$$
$$=-k^2U(x',z), \tag{14}$$

which shows that $U(x', z)$ is a solution to the 2-D Helmholtz equation, and Eq. (13) shows that the solution depends only on the value of U in the $z = 0$ plane. For the diffraction problem, the standard procedure is to use $U_0(x, 0)$ for $U(x, 0)$ in the transmissive parts of the screen and zero in the blocking parts. (Goodman loosely states that "Kirchhoff boundary conditions" are applied [1], but actually only the RS conditions are needed.) Thus, Fourier propagation and the RS integral are solutions to the same differential equation with the same boundary condition, hence must be the same function, a fact that is shown explicitly in the next section.

4.2 The RS Integral Derived via Fourier Propagation

Returning to Eq. (13), interchanging the order of integration and using the RS boundary condition,

$$U(x',z)=\int_x U_0(x,0)\int_\alpha \exp\left[ikz\left(\sqrt{1-\alpha^2}+\alpha\frac{x'-x}{z}\right)\right]d\left(\frac{\alpha}{\lambda}\right)dx$$
$$\equiv\int_x U_0(x,0)\int_\alpha \exp\left[ikzF(\alpha)\right]d\left(\frac{\alpha}{\lambda}\right)dx, \tag{15}$$

where the last line defines the function $F(\alpha)$. The α integral is done by the stationary phase method: the function $F(\alpha)$ is expanded in a Taylor series about the point α_0 at which its first derivative is zero. Using the notation $F_\alpha \equiv \partial F/\partial\alpha$, we require

$$F_\alpha(\alpha_0)=-\frac{\alpha_0}{\sqrt{1-\alpha_0^2}}+\frac{x'-x}{z}=0, \tag{16}$$

which can be solved for α_0,

$$\alpha_0=\frac{x'-x}{\sqrt{(x'-x)^2+z^2}}=\frac{x'-x}{r_{2D}}. \tag{17}$$

Therefore

$$\sqrt{1-\alpha_0^2}=\frac{z}{r_{2D}} \tag{18}$$

and

$$F(\alpha_0)=\sqrt{1-\alpha_0^2}+\alpha_0\frac{x'-x}{z}=\frac{r_{2D}}{z}. \tag{19}$$

The second derivative of $F(\alpha)$ at α_0 is

$$F_{\alpha\alpha}(\alpha_0) = -\frac{1}{\sqrt{1-\alpha_0^2}} - \frac{\alpha_0^2}{\sqrt{1-\alpha_0^2}^3} = -\frac{1}{\sqrt{1-\alpha_0^2}^3} = -\frac{r_{2D}^3}{z^3}. \tag{20}$$

Since the first-order term vanishes by construction, the Taylor series expansion through second order of $F(\alpha)$ about α_0 is

$$F(\alpha) \approx F(\alpha_0) + \frac{1}{2} F_{\alpha\alpha}(\alpha_0)(\alpha - \alpha_0)^2. \tag{21}$$

A standard Fresnel integral is now written in a form that will be useful below:

$$\int_{-\infty}^{\infty} \exp\left\{-i\left[\frac{\pi z}{\lambda} A(u-u_0)^2\right]\right\} d\left(\frac{u}{\lambda}\right) = \frac{1-i}{\sqrt{2}} \frac{1}{\sqrt{\lambda z A}}, \tag{22}$$

where A is positive definite.

The integral over α in Eq. (15) can now be evaluated. For ease of notation, drop the α_0 argument from $F_{\alpha\alpha}$ and observe that $F_{\alpha\alpha} = -|F_{\alpha\alpha}|$ ($F_{\alpha\alpha}$ is always negative). Eq. (21) is inserted into the exponent in Eq. (15) to obtain:

$$\begin{aligned}
&\exp\left[ikzF(\alpha_0)\right] \int_\alpha \exp\left\{ikz\left[-\frac{1}{2}\left|F_{\alpha\alpha}\right|(\alpha-\alpha_0)^2\right]\right\} d\left(\frac{\alpha}{\lambda}\right) \\
&\qquad = \exp\left(ikr_{2D}\right) \frac{1-i}{\sqrt{2}} \frac{1}{\sqrt{\lambda z\left|F_{\alpha\alpha}\right|}} \\
&\qquad = \exp\left(ikr_{2D}\right) \frac{1-i}{\sqrt{2}} \frac{1}{\sqrt{\lambda r_{2D}}} \frac{z}{r_{2D}}.
\end{aligned} \tag{23}$$

Therefore

$$\begin{aligned}
U(x', z) &= \int_x U_0(x,0) \exp\left(ikr_{2D}\right) \frac{1-i}{\sqrt{2}} \frac{1}{\sqrt{\lambda r_{2D}}} \frac{z}{r_{2D}} dx \\
&= \frac{1-i}{\sqrt{2\lambda}} \int_x U_0(x,0) \frac{\exp\left(ikr_{2D}\right)}{\sqrt{r_{2D}}} \cos \chi_{2D} dx.
\end{aligned} \tag{24}$$

Eq. (24) is the 2-D version of Eq. (3) and contains the 2-D Huygens' principle: each strip in the aperture acts as a source of cylindrical waves, for which amplitude falls off as $r_{2D}^{-1/2}$. The interested reader may want to make the normal approximations [$\cos\chi_{2D} \approx 1$, $r_{2D} \approx z + (x' - x)^2/2z$ in the exponent, $r_{2D} \approx z$ outside it] and show that Eq. (24) reduces to the standard form that is evaluated with the Cornu spiral.

Extending Eq. (24) to the 3-D problem will, first of all, add a dy to the integral. Keeping in mind that all the units of length must cancel out on the right side, inspection of Eq. (24) suggests that the effect of a 3-D calculation is to replace r_{2D} and χ_{2D} by r and χ, to replace cylindrical waves by spherical waves, and to square the factor outside the integral. This intuitive argument is confirmed in detail in Appendix B, with a result, Eq. (B14), that matches the RS form of the diffraction integral given in Eq. (3).

5. CONCLUSION

The fundamental flaw in the FK diffraction integral and the superiority of RS have been demonstrated with exact calculations of the intensity of Poisson's spot. Fourier propagation has been presented as an alternate means of deriving the diffraction integral. Compared to the usual approach via Green's theorem, this derivation has the advantage of rendering obvious the proper choice of boundary conditions. It has the disadvantage of requiring knowledge of Fourier propagation and, for the 3-D version, more complicated math, but the 2-D version is not bad and the generalization to 3-D by inspection is intuitively appealing.

As noted in the introduction, the argument has been confined to scalar wave theory, which will not be completely adequate for describing Poisson's spot in optics for points near the disk. (The contribution of those rays not polarized parallel to the diffracting edge should be multiplied by the sine of the angle between the polarization vector and the direction to the observation point.) But it should be entirely adequate for describing Poisson's spot in an acoustics experiment because acoustic waves are scalar (pressure) waves. Also, because the wavelength is much longer, diffraction phenomena can be more easily studied in an acoustics than in an optics laboratory. This experiment was done many years ago with somewhat equivocal results [8]. With modern equipment, it should not be particularly difficult to repeat, and could settle the conflict between RS and FK diffraction by direct measurement.

REFERENCES

1. J. W. Goodman, *Introduction to Fourier Optics* (John Wiley & Sons, New York, 1968), 1st ed., pp. 41-44 and Sec. 3.7.

2. A. Sommerfeld, *Lectures on Theoretical Physics*, v. 4: *Optics* (Academic Press, New York, 1954), pp. 198-201 and 211-212.

3. M. Born and E. Wolf, *Principles of Optics* (Pergamon Press, New York, 1970), pp. 381-382.

4. G.E. Sommargren and H.J. Weaver, "Diffraction of Light by an Opaque Sphere. 1: Description of the Diffraction Pattern," *Appl. Opt.* **29**, 4646-4657 (1990).

5. E.A. Hovenac, "Fresnel Diffraction by Spherical Obstacles," *Am. J. Phys.* **57**, 79-84 (1989).

6. J.E. Harvey, J.L. Forgham, and K. Von Bieren, "The Spot of Arago and Its Role in Wavefront Analysis," *Proceedings of SPIE*, Vol. 351 – *Wavefront Sensing*, N. Bareket and C.L. Koliopoulos, eds., Aug. 24 - 25, 1982, San Diego, California, pp. 2-9.

7. H. Osterberg and L.W. Smith, "Closed Solutions of Rayleigh's Diffraction Integral for Axial Points," *JOSA* **51**, 1050-1057 (1961).

8. H. Primakoff, M.J. Klein, J.B. Keller, and E.L. Carstensen, "Diffraction of Sound Around a Circular Disk," *J. Acoust. Soc. Am* **19**, 132-142 (1947).

Appendix A

THE SOMMERFELD LEMMA

Following Sommerfeld [A1] and Harvey et al. [A2], we perform a series of integrations by parts (only the first two are shown) to expand the integral of a function multiplied by a complex exponential in a series of terms:

$$\int_a^b f(x)\exp(ikx)\,dx = \frac{f(x)}{ik}\exp(ikx)\Bigg|_a^b - \frac{f'(x)}{(ik)^2}\exp(ikx)\Bigg|_a^b + \frac{1}{(ik)^2}\int_a^b f''(x)\exp(ikx)dx$$

$$\approx \frac{f(x)}{ik}\exp(ikx)\Bigg|_a^b, \qquad \text{(A1)}$$

where the approximation is justified if $f(x)$ is a slowly varying function, or, equivalently, in the limit $k \to \infty$ ($\lambda \to 0$). When applied to diffraction, the approximation in Eq. (A1) holds when the geometric dimensions of a problem are large compared to a wavelength of light, otherwise $f(x)$ may not be sufficiently slowly varying.

The integral in Eq. (A1) has exactly the form of the k-frequency coefficient of a Fourier series expansion of the function $f(x)$ over the interval (a, b). This shows that for large k, we need know only the values of $f(a)$ and $f(b)$ to find the value of the coefficient. I have searched more than a dozen Fourier series books and have not found this lemma in any of them.

REFERENCES

A1. A. Sommerfeld, *Lectures on Theoretical Physics*, v. 4: *Optics* (Academic Press, New York, 1954), pp. 198-201 and 211-212.

A2. J.E. Harvey, J.L. Forgham, and K. Von Bieren, “The Spot of Arago and Its Role in Wavefront Analysis,” *Proceedings of SPIE*, Vol. 351 – *Wavefront Sensing*, N. Bareket and C.L. Koliopoulos, eds., Aug. 24 - 25, 1982, San Diego, California, pp. 2-9.

Appendix B

DERIVING THE RS DIFFRACTION INTEGRAL VIA FOURIER PROPAGATION

Proceeding in analogy to Section 4.2, the 3-D version of Eq. (15) is

$$
\begin{aligned}
&U(x',\, y',\, z) \\
&\quad = \int_{x,y} U_0(x,y,0) \int_{\alpha,\beta} \exp\left[ikz\left(\sqrt{1-\alpha^2-\beta^2} + \alpha\,\frac{x'-x}{z} + \beta\,\frac{y'-y}{z}\right)\right] d\left(\frac{\alpha}{\lambda}\right) d\left(\frac{\beta}{\lambda}\right) dxdy \\
&\quad \equiv \int_{x,y} U_0(x,y,0) \int_{\alpha,\beta} \exp\left[ikzF(\alpha,\beta)\right] d\left(\frac{\alpha}{\lambda}\right) d\left(\frac{\beta}{\lambda}\right) dxdy.
\end{aligned}
\tag{B1}
$$

The function $F(\alpha, \beta)$ is expanded in a Taylor series about the point (α_0, β_0) at which its first derivatives are zero. Setting

$$
F_\alpha(\alpha_0,\beta_0) = -\frac{\alpha_0}{\sqrt{1-\alpha_0^2-\beta_0^2}} + \frac{x'-x}{z} = 0, \tag{B2}
$$

and similarly for β, leads to

$$
\alpha_0 = \frac{x'-x}{r}, \tag{B3}
$$

$$
\beta_0 = \frac{y'-y}{r}, \tag{B4}
$$

and

$$
\sqrt{1-\alpha_0^2-\beta_0^2} = \frac{z}{r}. \tag{B5}
$$

The second derivatives of $F(\alpha, \beta)$ at (α_0, β_0) are

$$
F_{\alpha\alpha}(\alpha_0,\, \beta_0) = -\frac{1}{\sqrt{1-\alpha_0^2-\beta_0^2}} - \frac{\alpha_0^2}{\sqrt{1-\alpha_0^2-\beta_0^2}^{\,3}} = -\frac{r[(x'-x)^2+z^2]}{z^3}, \tag{B6}
$$

$$F_{\beta\beta}(\alpha_0, \beta_0) = -\frac{r[(y'-y)^2 + z^2]}{z^3}, \tag{B7}$$

and

$$F_{\alpha\beta}(\alpha_0, \beta_0) = -\frac{\alpha_0\beta_0}{\sqrt{1-\alpha_0^2-\beta_0^2}^{\,3}} = -\frac{r(x'-x)(y'-y)}{z^3}. \tag{B8}$$

The following quantities will be needed below:

$$F(\alpha_0, \beta_0) = \frac{z}{r} + \frac{\alpha_0(x'-x)}{z} + \frac{\beta_0(y'-y)}{z} = \frac{r}{z} \tag{B9}$$

and

$$F_{\alpha\alpha}(\alpha_0, \beta_0)F_{\beta\beta}(\alpha_0, \beta_0) - F_{\alpha\beta}^2(\alpha_0, \beta_0) = \frac{r^4}{z^4}. \tag{B10}$$

The Taylor series expansion through second order of $F(\alpha, \beta)$ about the point (α_0, β_0) is

$$\begin{aligned} F(\alpha,\beta) \approx F(\alpha_0, \beta_0) &+ \frac{1}{2} F_{\alpha\alpha}(\alpha_0,\beta_0)(\alpha-\alpha_0)^2 + \frac{1}{2} F_{\beta\beta}(\alpha_0,\beta_0)(\beta-\beta_0)^2 \\ &+ F_{\alpha\beta}(\alpha_0,\beta_0)(\alpha-\alpha_0)(\beta-\beta_0). \end{aligned} \tag{B11}$$

The α and β integrals in Eq. (B1) can now be evaluated. Observe that $F_{\alpha\alpha} = -|F_{\alpha\alpha}|$ ($F_{\alpha\alpha}$ is always negative) and, for ease of notation, drop the (α_0, β_0) argument from the quantities $F_{\alpha\alpha}$, $F_{\beta\beta}$, and $F_{\alpha\beta}$. The first, second, and fourth terms in Eq. (B11) are inserted into the exponent in Eq. (B1), and the integral over α is evaluated by completing the square in the exponent (the symbol $\pm$ in Eq. (B12) means add <u>and</u> subtract, not add <u>or</u> subtract):

$$\begin{aligned} &\exp\left[ikzF(\alpha_0, \beta_0)\right] \int_\alpha \exp\left\{ikz\left[-\frac{1}{2}\left|F_{\alpha\alpha}\right|(\alpha-\alpha_0)^2 + F_{\alpha\beta}(\alpha-\alpha_0)(\beta-\beta_0)\right]\right\} d\left(\frac{\alpha}{\lambda}\right) \\ &= \exp(ikr) \int_\alpha \exp\left\{-\frac{ikz}{2}\left|F_{\alpha\alpha}\right|\left[(\alpha-\alpha_0)^2 - 2\frac{F_{\alpha\beta}}{|F_{\alpha\alpha}|}(\alpha-\alpha_0)(\beta-\beta_0) \pm \frac{F_{\alpha\beta}^2}{|F_{\alpha\alpha}|^2}(\beta-\beta_0)^2\right]\right\} d\left(\frac{\alpha}{\lambda}\right) \\ &= \exp(ikr)\exp\left[\frac{ikz}{2}\frac{F_{\alpha\beta}^2}{|F_{\alpha\alpha}|}(\beta-\beta_0)^2\right] \int_\alpha \exp\left\{-\frac{ikz}{2}|F_{\alpha\alpha}|\left[\alpha-\alpha_0-\frac{F_{\alpha\beta}}{|F_{\alpha\alpha}|}(\beta-\beta_0)\right]^2\right\} d\left(\frac{\alpha}{\lambda}\right) \\ &= \exp(ikr)\exp\left[\frac{ikz}{2}\frac{F_{\alpha\beta}^2}{|F_{\alpha\alpha}|}(\beta-\beta_0)^2\right] \frac{1-i}{\sqrt{2}} \frac{1}{\sqrt{\lambda z|F_{\alpha\alpha}|}}. \end{aligned} \tag{B12}$$

The integral over β in Eq. (B1) can now be carried out by adding the third term in Eq. (B11) to the exponent in Eq. (B12), and again using $F_{\alpha\alpha} = -|F_{\alpha\alpha}|$:

$$\begin{aligned}
&\frac{1-i}{\sqrt{2}} \frac{1}{\sqrt{\lambda z |F_{\alpha\alpha}|}} \exp(ikr) \int_{\beta} \exp\left(\frac{ikz}{2} \frac{F_{\alpha\beta}^2}{|F_{\alpha\alpha}|} (\beta-\beta_0)^2 + \frac{ikz}{2} F_{\beta\beta}(\beta-\beta_0)^2 \right) d\left(\frac{\beta}{\lambda}\right) \\
&= \frac{1-i}{\sqrt{2}} \frac{1}{\sqrt{\lambda z |F_{\alpha\alpha}|}} \exp(ikr) \int_{\beta} \exp\left[-\frac{ikz}{2} \left(\frac{F_{\alpha\alpha}F_{\beta\beta} - F_{\alpha\beta}^2}{|F_{\alpha\alpha}|} \right) (\beta-\beta_0)^2 \right] d\left(\frac{\beta}{\lambda}\right) \\
&= \frac{1-i}{\sqrt{2}} \frac{1}{\sqrt{\lambda z |F_{\alpha\alpha}|}} \exp(ikr) \frac{1-i}{\sqrt{2}} \sqrt{\frac{|F_{\alpha\alpha}|}{\lambda z (F_{\alpha\alpha}F_{\beta\beta} - F_{\alpha\beta}^2)}} \\
&= -\frac{iz}{\lambda r^2} \exp(ikr). \qquad \text{(B13)}
\end{aligned}$$

Eq. (B1) can now be written in the desired form:

$$\begin{aligned}
U(x', y', z) &= -\frac{i}{\lambda} \int_{x,y} U_0(x, y, 0) \frac{z \exp(ikr)}{r^2} dxdy \\
&= -\frac{i}{\lambda} \int_{x,y} U_0(x, y, 0) \frac{\exp(ikr)}{r} \cos\chi \, dxdy. \qquad \text{(B14)}
\end{aligned}$$